On a Generalization of Ramsey Theory

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Abstract

Classical Multi-color Ramsey Theory pertains to the existence of monochromatic subsets of structured multicolored sets such that the subsets have a given property or structure. This paper examines a generalization of Ramsey theory that allows the subsets to have specified groupings of colors. By allowing more than one color in subsets, the corresponding minimal sets for finite cases tend to be smaller. The author proposes poly-chromatic Ramsey Theory for the most general case and tuple-chromatic Ramsey Theory for the case in which the same number of colors is the maximum allowed for all of the subsets.

1 Introduction

Ramsey Theory includes such theorems as Ramsey's theorem, Van der Waerden's theorem, and Rado's theorem. In each of these theorems, a set is colored with 2 or more colors. These theorems affirmatively answer the question of whether a monochromatic subset exists with a given property. While these theorems address the question of existence of a required subset, a great amount of work has gone into determining lower bounds on the sizes of sets containing monochrome subsets of a given size. Radziszowski's Dynamic Survey [R] presents many of these lower bounds for Ramsey numbers.

Poly-chromatic Ramsey Theory considers subsets with a given property that have specified groupings of colors. If all these subsets of colors contain exactly q colors, then this is q-tuple-chromatic Ramsey Theory. In the particular case in which all the subsets contain exactly one color, then this is the same as classical Ramsey Theory.

Since a monochromatic subset also qualifies as a subset with at most some other number of colors, all of the existence proofs in Ramsey Theory provide the same affirmative answers for poly-chromatic Ramsey Theory. The lower bounds for satisfying subsets, however, are generally lower than their standard Ramsey Theory counterparts.

2 Notation

In order to discuss these new numbers, additional notation is required. Ramsey numbers are frequently written as follows [R].

$$R(G_1, ..., G_r)$$

This is informally defined as the smallest integer $n \in \mathbb{N}$ such that for any ordered r-coloring of the edges of the complete graph K_n, there exists one of the colors, say i, and a subgraph of K_n that is isomorphic to G_i and with only edges of color i. Many generalizations of this notation exist.

The following definition helps in defining poly-chromatic Ramsey and Van der Waerden numbers. It is simply used to assign a unique numeric value to each element of the power set of the set $\{1, ..., r\}$.

Definition 1. *Let $\mathcal{P}(A)$ be the power set of set A. For any natural number $r \in \mathbb{N}$, define the natural ordering on $\mathcal{P}(\{1, ..., r\})$ as the function $h_r : \mathcal{P}(\{1, ..., r\}) \to \{0, ..., 2^r - 1\}$ defined as*

$$h_r(X) := \sum_{i \in X} 2^{i-1}, \text{ for any } X \in \mathcal{P}(\{1, ..., r\}) \tag{1}$$

The function h_r bijective, so $h_r^{-1} : \{0, ..., 2^r - 1\} \to \mathcal{P}(\{1, ..., r\})$ is well-defined.

The next definition makes colorings, and in particular edge colorings a little more precise.

Definition 2. *For any $r \in \mathbb{N}$, an r-**coloring** of a set A is a function $c : A \to \{1, ..., r\}$. For a graph with edge set E, an r-**edge-coloring** is a function $c : E \to \{1, ..., r\}$. Furthermore, if $X \in \mathcal{P}(\{1, ..., r\})$, then an X-**coloring** of a set A is a function $c : A \to X$. For a graph with edge set E, an X-**edge-coloring** is a function $c : E \to X$.*

Poly-chromatic Ramsey numbers allow various groupings of colors to be considered for containing a specified subgraph.

Definition 3. *For any $r \in \mathbb{N}$, let $\mathcal{G} = \{G_0, ..., G_{2^r-1}\}$ be an ordered set of 2^r non-directed graphs, some of which may be the empty graph, and such that if*

$$\mathcal{I} = \{i \in \{0, ..., 2^r - 1\} | G_i \neq \text{the empty graph}\},$$

then $\bigcup_{i \in \mathcal{I}} h_r^{-1}(i) = \{1, ..., r\}$.

Define the function

$$pcR(r; G_0, ..., G_{2^r-1}) \tag{2}$$

*as the minimum number $n \in \mathbb{N}$ such that for any r-edge-coloring: c of the complete graph of size n: K_n, there exists at least one number $i \in \{0, ..., 2^r - 1\}$, such that G_i is not the empty graph and there is a subgraph S of K_n that is isomorphic to G_i, and such that S is $h_r^{-1}(i)$-edge-colored by the restriction of c to the edges of S. The value of $pcR(r; G_0, ..., G_{2^r-1})$ is the **poly-chromatic Ramsey number** in r colors on the ordered set of graphs $\mathcal{G}$.*

The following definition is merely a change of notation in the case where the only non-empty subgraphs are those with indices corresponding to q colors in the natural ordering of the power set of the edge colors.

Definition 4. *If, in the definition of poly-chromatic Ramsey numbers, the non-empty graphs of $\mathcal{G}$ are all those graphs G_i, such that $\#(h_r^{-1}(i)) = q$ for a given $q \in \mathbb{N}$, then the notation only needs to include graphs with indices that represent q colors. The revised notation has a new function name, tcR, to reduce confusion. It is written as*

$$tcR(r, q; G_{x_1}, G_{x_2}, ..., G_{x_{\binom{r}{q}}}) \tag{3}$$

*where the x_i are the elements of $\{0, ..., 2^r - 1\}$ such that $\#(h^{-1}(x_i)) = q$, taken in order. The value of $tcR(r, q; G_{x_1}, G_{x_2}, ..., G_{x_{\binom{r}{q}}})$ is the **tuple-chromatic Ramsey number** in q out of r colors on the set of graphs $\{G_{x_1}, G_{x_2}, ..., G_{x_{\binom{r}{q}}}\}$.*

If $q = 1$, then **tuple-chromatic Ramsey numbers** are the same as multi-color Ramsey numbers. By definition, we have

$$R(G_{2^0}, G_{2^1}, ..., G_{2^{r-1}}) = tcR(r, 1; G_{2^0}, G_{2^1}, ..., G_{2^{r-1}}) \tag{4}$$

As with Ramsey numbers, if all the non-empty graphs form a set of complete graphs K_{k_i}, then we can write:

$$pcR(r; k_0, k_1, ..., k_{2^r-1}) \tag{5}$$

where $k_i = 0$ implies an empty graph for that value of i.

Similarly, we can write the tuple-chromatic version in the simplified notation:

$$tcR(r, q; k_{x_1}, k_{x_2}, ..., k_{x_{\binom{r}{q}}}) \tag{6}$$

For tuple-chromatic Ramsey numbers on complete graphs all having the same size k, we can write:

$$tcR(r, q; k) \tag{7}$$

If $q = 1$, these are the same as the diagonal case multi-color Ramsey numbers:

$$R_r(k) = tcR(r, 1; k) \tag{8}$$

Van der Waerden numbers can also be extended to poly-chromatic and tuple-chromatic variants. Van der Waerden numbers are typically defined as follows [LR].

$$W(k), \text{ for } k \in \mathbb{N} \tag{9}$$

is the number $n \in \mathbb{N}$ such that any two-coloring of the set $[n] = \{1, ..., n\}$, there exists a monochrome subset of size k with terms that are in an arithmetic progression. Multi-colored Van der Waerden numbers are typically defined as

$$W(r; k), \text{ for } r, k \in \mathbb{N} \tag{10}$$

is the number n such that any r-coloring of the set $[n] = \{1, ..., n\}$, there exists a monochrome subset of size k with terms that are in an arithmetic progression. If we consider modifying the requirement that any color have an arithmetic progression of length k to one in which the length of the required arithmetic subsequence be k_i for the i-th color, then we have:

$$W(r; k_1, ..., k_r) \tag{11}$$

The polychromatic variant of (11) is as follows.

Definition 5. *For any $r \in \mathbb{N}$, let $\mathcal{K} = \{k_0, ..., k_{2^r-1}\}$ be an ordered set of non-negative integers, such that if*

$$\mathcal{I} = \{i \in \{0, ..., 2^r - 1\} | k_i \neq 0\},$$

then $\bigcup_{i \in \mathcal{I}} h_r^{-1}(i) = \{1, ..., r\}$.

$$pcW(r; k_0, ..., k_{2^r-1}) \tag{12}$$

is defined to be the minimum number $n \in \mathbb{N}$ such that if the set $\{1, ..., n\}$ is r-colored by the function c, then there exists an $i \in \mathcal{I}$ and a subset S of $\{1, ..., n\}$ with the terms of a finite arithmetic sequence of length k_i such that S is $h_r^{-1}(i)$-colored by the restriction of c to the elements of S. The value of $pcW(r; k_0, ..., k_{2^r-1})$ is the ***poly-chromatic Van der Waerden number*** *for r colors for the numbers $\{k_0, ..., k_{2^r-1}\}$.*

The tuple-chromatic notation follows similarly to that of tuple-chromatic Ramsey numbers.

Definition 6. *If, in the definition of poly-chromatic Van der Waerden numbers, the non-zero numbers of $\mathcal{K}$ are all those numbers k_i, such that $\#(h_r^{-1}(i)) = q$ for a given $q \in \mathbb{N}$, then the notation only needs to include numbers with indices that represent q colors. The revised notation has a new function name, tcW, to reduce confusion. It is written as*

$$tcW(r, q; k_{x_1}, k_{x_2}, ..., k_{x_{\binom{r}{q}}}) \tag{13}$$

*where the x_i are the elements of $\{0, ..., 2^r - 1\}$ such that $\#(h^{-1}(x_i)) = q$, taken in order. The value of $tcW(r, q; k_{x_1}, ..., k_{x_{\binom{r}{q}}})$ is the **tuple-chromatic Van der Waerden number** in q out of r colors for the numbers $\{k_{x_1}, ..., k_{x_{\binom{r}{q}}}\}$.*

If all of the k_{x_i} have the same value k, then the notation for the tuple-chromatic Van der Waerden number can be simplified to

$$tcW(r, q; k) = tcW(r, q; k, k, ..., k) \text{ where the right hand expression has } \tbinom{r}{q} \text{ copies of } k. \tag{14}$$

3 Basic Results

As with classic Ramsey numbers, computing actual values for tuple-chromatic Ramsey numbers is very difficult. We can find some inequalities that help put bounds on their size.

For tuple-chromatic Ramsey numbers of the form $tcR(r, q; k)$ we have the following inequalities.

Theorem 1.

$$\begin{aligned}&\text{For any } t \in \mathbb{N},\\ &tcR(t{\cdot}r, t{\cdot}q; k) \leq tcR(r, q; k)\end{aligned} \tag{15}$$

Proof. Suppose that we have a complete graph G of size $tcR(r, q; k)$ with $t{\cdot}r$ edge colors numbered from 0 to $t{\cdot}r - 1$. Create a new coloring of the edges of G by taking the original edge colors modulo r. Since the new coloring has r colors, we can find a complete subgraph H of size k in q colors. Each of the q colors correspond to at most t colors from the original coloring, so the number of edge colors of H in the original coloring is at most $t{\cdot}q$. The graph G is guaranteed to have a subgraph in at most $t{\cdot}q$ colors, therefore $tcR(t{\cdot}r, t{\cdot}q; k)$ must be less than or equal to the size of G: $tcR(r, q; k)$. □

Essentially the same proof works for tuple-chromatic Van der Waerden numbers.

Theorem 2.

$$\begin{aligned}&\textit{For any } t \in \mathbb{N},\\ &tcW(t{\cdot}r, t{\cdot}q; k) \leq tcW(r, q; k)\end{aligned} \tag{16}$$

Proof. Suppose that we have a coloring of the positive integers from 1 to $tcW(r, q; k)$ in $t{\cdot}r$ colors. Create a new coloring of the numbers by taking the original coloring modulo r. Since the new coloring has r colors, we can find a subset of the numbers that form an arithmetic progression of length k and having at most q colors. The q colors correspond to at most $t{\cdot}q$ colors in the original coloring. Therefore the inequality applies. □

An even simpler argument shows that increasing the value of q decreases the corresponding tuple-chromatic number.

Theorem 3.

$$\begin{aligned}&\textit{For any } q' \in \{q, ..., r-1\},\\ &tcR(r, q'; k) \leq tcR(r, q; k)\end{aligned} \tag{17}$$

Proof. Suppose that we have a complete graph of size $tcR(r, q; k)$ with r edge colors. By definition, we must have a complete subgraph with edges colored using at most q colors. Because $q \leq q'$, this subgraph also has at most q' edge colors.

□

The next theorem provides a way of generating bounds on tuple-chromatic Ramsey numbers based on other tuple-chromatic Ramsey numbers.

Theorem 4.

$$\begin{aligned}&\textit{For } t \in \{1, ..., r-q-1\},\\ &tcR(r, q+t; k+1) \leq \left\lfloor \tfrac{r}{t}\ tcR(r, q; k) \right\rfloor + 1\end{aligned} \tag{18}$$

Proof. Suppose that we have a complete graph G of size

$$n = \left\lfloor \frac{r}{t}\ tcR(r, q; k) \right\rfloor + 1 \tag{19}$$

with r edge colors. Choose a vertex V. By a generalization of the *Pigeonhole Principle*, there must exist t colors such that

$$m = \left\lceil \frac{t}{r}\ (n-1) \right\rceil$$

edges with those colors are coincident with V. Substituting the value of n from (19), we get

$$m \geq tcR(r, q; k)$$

.

Consider the complete graph formed by the m vertices connected to V. There must be a complete subgraph H of size k with at most q edge colors within that graph. Since the subgraph is also connected to V in at most t edge colors, the complete graph J that includes V and all of H has at most $q + t$ edge colors. The graph J has $k + 1$ vertices, therefore

$$tcR(r, q + t, k + 1) \leq n$$

□

Table 1: Tuple-chromatic Ramsey Numbers

r,q	k=2	k=3	k=4	k=5	k=6	k=7
3,2	2	5	11+	17+	24+	32+
4,2	2	5	13+	25+	39+	55+
4,3	2	3	8+	11+	16+	21+
5,2	2	6	15+	31+	58+	107+
5,3	2	8+	14+			

4 Tuple-chromatic Ramsey Numbers

A C++ program was used to find lower bounds on tuple-chromatic Ramsey numbers. The source code of the C++ program can be found at the following location ... The program creates a random colored complete graph and then keeps adjusting it until it finds a graph with a lower number for the size of the maximal complete subgraph in q colors. By combining the results of multiple runs, a lower bound can be found for a tuple-chromatic Ramsey number. A complete graph of size n in r colors such that the maximum complete graph in q colors is $k-1$ is called a witness for $tcR(r, q; k)$. A witness of size n for $tcR(r, q; k)$ implies that $n + 1 \leq tcR(r, q; k)$.

The following table shows lower bounds found using the program. Numbers without the + sign are easily proven. Numbers with the + sign are lower bounds with witnesses.

5 Tuple-chromatic Van der Waerden Numbers

A C++ program was used to compute actual values for many small tuple-chromatic Van der Waerden numbers $tcW(r, q; k)$. It used brute-force to try a sufficient set of r-colorings of the numbers starting with 1. Any time a coloring admitted an arithmetic subsequence of length k that used only q colors, it backtracked. Once the sufficient number of possibilities were tried, it would print the longest sequence of numbers with an r-coloring that didn't include any arithmetic sub-sequences with a q-coloring.

The program didn't need to check every possible coloring, because only one permutation of colors was necessary to prove that a maximal witness was found. In order to limit the testing to one permutation, the order of the first appearance of each color was always ascending.

Further improvements were made by creating future block lists as subsequences were encountered. Specifically, if a subsequence had $k-1$ elements in q colors, but the kth element back would make $q+1$ colors, then the program looked forward and blocked the use of any of the used q colors in the future kth position. It also had to keep track of when the blocks were still in effect, so that it didn't block a trial that it should have checked.

For some values, the program could not reach completion. In these cases, the longest sequence was used to determine a lower bound for the value. These numbers appear with a + sign after them.

The values on the diagonals tend to be the easiest to compute. In particular, values for which $k = q + 1$ and values for which r, q and k are all close. This suggests that a sequence of some length may be found in $tcW(r, r-1; r)$. The values of this sequence are (starting with r=2).

3, 7, 7, 21, 11, 43, 15, 25, 19, 111, 23, 157, 27, 43, ...

It turns out that the terms of this sequence are determined by a very simple expression.

Theorem 5. *If $r \in \mathbb{N}, r > 1$, and p is the smallest prime factor of r, then*

$$tcW(r, r-1; r) = (r-1)p + 1 \tag{20}$$

Proof. Let $r \in \mathbb{N}, r > 1$, and p is the smallest prime factor of r. The proof proceeds to construct the longest possible r-colored sequence that does not have an arithmetic subsequence of length r in $r-1$ or less colors and then conclude that the value of $tcW(r, r-1; r)$ must be one more than the longest such sequence. Let S be that maximal sequence with terms $S = (s_0, s_1, ..., s_{n-1})$ where n is the length of S.

Since we must avoid sequences of length r in $r-1$ or less colors, the first r terms of S must include all r colors. Without loss of generality, we can choose the colors $\{0, ..., r-1\}$ in order, for the first r terms as permutations of the colors do not matter. Likewise, every r consecutive terms of S must contain all r colors. Therefore, the sequence of colors repeats

Table 2: Tuple-chromatic Van der Waerden Numbers

r,q	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10
3,2	7	13	25	51	105+			
4,2	7	23	59+					
4,3	3	7	22	29	46+			
5,2	11	36	100+					
5,3	3	16	25	44+				
5,4	3	4	21	26	45+			
6,2	11	51+						
6,3	3	16	45+					
6,4	3	4	21	37	43+			
6,5	3	4	5	11	44	51+		
7,2	15	70+	100+					
7,3	3	22	45+					
7,4	3	4	29	39+				
7,5	3	4	5	36	44+			
7,6	3	4	5	6	43	50	57+	
8,2	15	84+						
8,3	3	22	53+					
8,4	3	4	29	38+				
8,5	3	4	5	36	43+			
8,6	3	4	5	6	43	50+		
8,7	3	4	5	6	7	15	25	32+
9,2	19	84+						
9,3	3	24	61+					
9,4	3	4	29	38+				
9,5	3	4	5	36	43+			
9,6	3	4	5	6	43	50+		
9,7	3	4	5	6	7	22	30	100+
9,8	3	4	5	6	7	8	25	28
10,2	19	84+						
10,3	3	25	48+					
10,4	3	4	31	38+				
10,5	3	4	5	36+				
10,6	3	4	5	6	43+			
10,7	3	4	5	6	7	22	89+	
10,8	3	4	5	6	7	8	25	100+
10,9	3	4	5	6	7	8	9	19

itself for the duration of S. With this arrangement of colors, if a is the color of s_i, then $a \equiv i(\text{modulo } r)$.

If $a \in \mathbb{N}$ is relatively prime to r and $b \in \mathbb{N}$ is any starting value, then the possible subsequence $(b, b+a, b+2a, ..., b+(r-1)a)$ taken modulo r contains all r colors. On the other hand, if a is not relatively prime to r, then it must contain repeated colors.

The smallest number that is not relatively prime to r, aside from 1 is p. Starting with an offset of 0, and incrementing by p, the colors must repeat before reaching all r terms of the subsequence. This subsequence reaches r terms at $(r-1)p+1$. Therefore, the length of the longest sequence without an $(r-1)$-colored arithmetic subsequence is $(r-1)p$ and $tcW(r, r-1; r) = (r-1)p+1$.

$\square$

We can also look at the sequence formed by terms of the form $tcW(r, r-1; r+1)$. The values of this sequence are (starting with r=2).

9, 13, 22, 26, 44, 50, 25, 28, 111+, 122, 157+, 170+, ...

At this stage I have not discovered a pattern to these numbers, but I feel that a pattern found in this sequence could lead to a breakthrough in Van der Waerden numbers.

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